Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Probability III(B3)

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Max. points : 50. **Time Limit : 3 hours.**

Give complete proofs. Please cite/quote appropriate results from class or assignments properly. You are also allowed to use results from other problems in the question paper.

Unless mentioned all state spaces are finite.

1. For an HMC $\{X_n\}$, let $f_{i,j} = \mathbb{P}_i(T_j < \infty)$ and $r_{i,j} = \mathbb{E}_i(N_j)$ where T_j is the return time to j and N_j is the number of visits (including X_0) to j . Show that the following hold :

   (a) For $m \geq 1$, $\mathbb{P}_j(N_j = m) = f_{j,j}^{m-1}(1 - f_{j,j})$ (**5**)

   (b) For $i \neq j$,

   $$\mathbb{P}_i(N_j = 0) = 1 - f_{i,j} \ ; \ \mathbb{P}_i(N_j = m) = f_{i,j} f_{j,j}^{m-1}(1 - f_{j,j}), \ m \geq 1. \ (\mathbf{5})$$

2. Let μ_j, ν_j be probability distributions on $E_j, j = 1, \ldots, d$. Define probability distributions on $E := \prod_{j=1}^d E_j$ as $\mu = \prod_j \mu_j, \nu = \prod_j \nu_j$. Show that $d_{TV}(\mu, \nu) \leq \sum_{j=1}^d d_{TV}(\mu_j, \nu_j)$. (**5**)

3. An HMC on E is transitive if $\forall x, y \in E$ there exists a bijection $\phi := \phi_{(x,y)} : E \to E$ such that

   $$\phi(x) = y, \ P(z, w) = P(\phi(z), \phi(w)), \ \forall z, w \in E.$$

   Let P be the transition matrix of a transitive HMC. Let $\hat{P}$ be the transition matrix of the time-reversed Markov chain . Show the following.

   (a) Show that the uniform distribution is a stationary distribution for transitive HMC. (**5**)

   (b) Show that when the HMC is transitive, then time-reversed Markov chain is also transitive. (**5**)

(c) Let $P, \hat{P}$ be as defined above and π, the uniform distribution on E. Let $x \in E$. Show that for all $t \geq 0$,

$$d_{TV}(P^t(x,.), \pi(.)) = d_{TV}(\hat{P}^t(x,.), \pi(.)). \ (\mathbf{10})$$

4. Let X_n be the reflected random walk on $\mathbb{N}$ with transition probabilities as follows : For $i > 0$, the random walk goes from i to $i+1$ with probability p, it goes from i to $(i-1)^+$ with probability q and with probability r, it stays at i. Of course, $p, q, r \geq 0$ and $p+q+r = 1$. Show that $P(X_n \geq c(p-q)n) \to 1$ for any $c < 1$. (**10**)

5. *Moran model with selection :* Let there be N individuals of two types - say Type I and Type II. Type I individuals have fitness level $\phi \in [1, \infty)$ and Type II individuals have fitness level 1. The population evolves as follows :

   - Given the population in generation n, an individual is chosen at random with probability proportional to its fitness level. The individual gives birth to an offspring of same type. The offspring replaces a randomly (i.e., uniformly at random) chosen individual from the exisiting population (i.e., the population at generation n), so that the total population remains constant.

   Let X_n be the number of Type I individuals in generation n. What are the absorption states and the probability of absorption into these states of the HMC X_n ? (**15**)

6. Let P be the transition matrix of a HMC on E with π as its stationary distribution. Let $f : E \to \mathbb{R}$. Show that $VAR_\pi(P^t f) \leq \lambda_*^{2t} VAR_\pi(f)$. (**15**)

   Recall that $\lambda_* := \max\{|\lambda| : \lambda \neq 1, \lambda \, \text{is an eigenvalue of P}\}$,
   $\mathbb{E}_\pi(f) :=< f, \mathbf{1} >_\pi, VAR_\pi(f) := \mathbb{E}_\pi([f - \mathbb{E}_\pi(f)]^2)$.

   (**Hint:** Compute $E_\pi(P^t f)$ and use spectral decomposition for $P^t f$.)